

Chapter 1

Exotic baryon resonances in the Skyrme model

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We outline how one can understand the Skyrme model from the modern perspective. We review the quantization of the $SU(3)$ rotations of the Skyrmion, leading to the exotic baryons that cannot be made of three quarks. It is shown that in the limit of large number of colours the lowest-mass exotic baryons can be studied from the kaon-Skyrmion scattering amplitudes, an approach known after Callan and Klebanov. We follow this approach and find, both analytically and numerically, a strong Θ^+ resonance in the scattering amplitude that is traced to the rotational mode. The Skyrme model *does* predict an exotic resonance Θ^+ but grossly overestimates the width. To understand better the factors affecting the width, it is computed by several methods giving, however, identical results. In particular, we show that insofar as the width is small, it can be found from the transition axial constant. The physics leading to a narrow Θ^+ resonance is briefly reviewed and affirmed.

1.1. How to understand the Skyrme model

It is astounding that Skyrme had suggested his model¹ as early as in 1961 before it has been generally accepted that pions are (pseudo) Goldstone bosons associated with the spontaneous breaking of chiral symmetry, and of course long before Quantum Chromodynamics (QCD) has been put forward as the microscopic theory of strong interactions.

The revival of the Skyrme idea in 1983 is due to Witten² who explained the *raison d'être* of the Skyrme model from the viewpoint of QCD. In the chiral limit when the light quark masses m_u, m_d, m_s tend to zero, such that the octet of the pseudoscalar mesons π, K, η become nearly massless (pseudo) Goldstone bosons, they are the lightest degrees of freedom of QCD. The effective chiral Lagrangian ($E\chi L$) for pseudoscalar mesons, understood as an infinite expansion in the derivatives of the pseudoscalar (or chiral) fields, encodes, in principle, full information about QCD. The famous two-term Skyrme Lagrangian can be understood as a low-energy truncation of this infinite series. Witten has added an important four-

derivative Wess–Zumino term³ to the original Skyrme Lagrangian and pointed out that the overall coefficient in front of the $E\chi L$ is proportional to the number of quark colours N_c .

Probably most important, Witten has shown that Skyrme's original idea of getting the nucleon as a soliton of the $E\chi L$ is justified in the limit of large N_c (since quantum corrections to a classical saddle point die out as $1/N_c$) and that the 'Skyrmion' gets correct quantum numbers upon quantization of its rotations in ordinary and flavour spaces. Namely, if one restricts oneself to two light flavours u, d , the lowest rotational states of a Skyrmion are the nucleon with spin $J = \frac{1}{2}$ and isospin $T = \frac{1}{2}$ and the Δ resonance with $J = \frac{3}{2}$ and $T = \frac{3}{2}$. For three light flavours u, d, s the lowest rotational state is the $SU(3)$ octet with spin $\frac{1}{2}$ and the next is the decuplet with spin $\frac{3}{2}$, in full accordance with reality. The statement appeared in Witten's 'note added in proof' without a derivation but a number of authors⁴ have derived the result (it is reproduced in Section 2). Almost all of those authors noticed that formally the next rotational excitation of the Skyrmion is an exotic baryon *antidecuplet*, again with spin $\frac{1}{2}$, however few took it seriously. It was only after the publication of Ref.⁵ where it was predicted that the lightest member of the antidecuplet, the Θ^+ baryon, must be light and narrow, that a considerable experimental and theoretical interest in the exotic baryons has been aroused.

Soon after Witten's work it has been realized that it is possible to bring the Skyrme model and the Skyrmion even closer to QCD and to the more customary language of constituent quarks. It has been first noticed^{6–8} that a simple chiral-invariant Lagrangian for massive (constituent) quarks Q interacting with the octet chiral field $\pi^A (A = 1, \dots, 8)$,

$$\mathcal{L} = \bar{Q} \left(\not{\partial} - M e^{\frac{i\pi^A \lambda^A \gamma_5}{F_\pi}} \right) Q, \quad \pi^A = \pi, K, \eta, \quad (1.1)$$

induces, via a quark loop in the external pseudoscalar fields (see Fig. 1), the $E\chi L$ whose lowest-derivative terms coincide with the Skyrme Lagrangian, including automatically the Wess–Zumino term, with the correct coefficient!

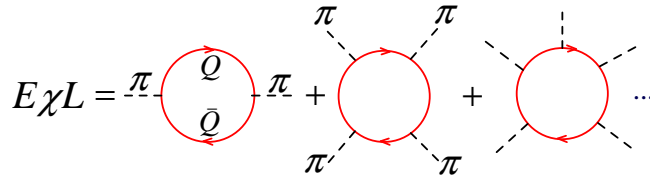


Fig. 1.1. The effective chiral lagrangian ($E\chi L$) is the quark loop in the external chiral field, or the determinant of the Dirac operator (1.1). Its real part is the kinetic energy term for pions, the Skyrme term and, generally, an infinite series in derivatives of the chiral field. Its imaginary part is the Wess–Zumino term, plus also an infinite series in derivatives.^{6,7,9}

A step in the same direction, namely in bringing the Skyrme model closer to the language of quarks, has been made in the chiral bag model by Brown, Rho and

collaborators,¹⁰ for a review see Ref.¹¹

In fact, Eq. (1.1) can be derived in the instanton liquid model for the spontaneous chiral symmetry breaking⁸ where a dynamical momentum-dependent quark mass $M(p)$ is generated as an originally massless quark propagates through the random ensemble of instantons and anti-instantons, each time flipping its helicity. The low-energy quark Lagrangian (1.1) is generally speaking nonlocal which provides a natural ultraviolet cutoff. At low momenta, however, one can treat the dynamical mass as a constant $M(0) \approx 350 \text{ MeV}$.⁸

It is implied that all gluon degrees of freedom, perturbative and not, are integrated out when one comes to the effective low-energy quark Lagrangian of the type given by Eq. (1.1). Important, one does not need to add explicitly, say, the kinetic energy term for pions to Eq. (1.1) (as several authors have originally suggested^{12–14}) since the pion is a $Q\bar{Q}$ state itself and it propagates through quark loops, as exhibited in the first graph in Fig. 1.

Understanding the quark origin of the E χ L it becomes possible to formulate what is the Skyrmion in terms of quarks and demystify the famous prescription of the Skyrme model that a chiral soliton with a topological (or winding) number equal to unity, is in fact a fermion.

To that end, one looks for a trial chiral field capable of binding constituent quarks. Let there be such a field $\pi(\mathbf{x})$ that creates a bound-state level for “valence” quarks, E_{val} . Actually, one can put N_c quarks at that level in the antisymmetric colour state, as the chiral field is colour-blind. The energy penalty for creating the trial field is given by the same Lagrangian (1.1). It is the aggregate energy of the negative-energy Dirac sea of quarks distorted by the trial field, E_{sea} ; it should be also multiplied by N_c since all negative-energy levels should be occupied and they are N_c -fold degenerate in colour. Therefore, the full energy of a state with baryon number unity and made of N_c quarks, is a sum of two functionals,^{9,15}

$$\mathcal{M}_N = N_c (E_{\text{val}}[\pi(\mathbf{x})] + E_{\text{sea}}[\pi(\mathbf{x})]). \quad (1.2)$$

Schematically it is shown in Fig. 2. The self-consistent (or mean) pion field binding quarks is the one minimizing the nucleon mass. Quantum fluctuations about it are suppressed insofar as N_c is large. The condition that the winding number of the trial field is unity needs to be imposed to get a deeply bound state, that is to guarantee that the baryon number is unity.⁹ The Skyrmion is, thus, nothing but the **mean chiral field binding quarks in a baryon**.

This model of baryons, called the *Chiral Quark Soliton Model* or the *Relativistic Mean Field Approximation*, apparently interpolates between the nonrelativistic constituent quark model and the Skyrme model, making sense and showing the limitations of both. Indeed, if the mean field happens to be small, the valence level is shallow, the Dirac sea is weakly distorted, and there are few antiquarks. In this case the model reproduces the well-known spin and space quark wave functions of the nonrelativistic models for baryons.¹⁶ If, on the contrary, the mean field happens

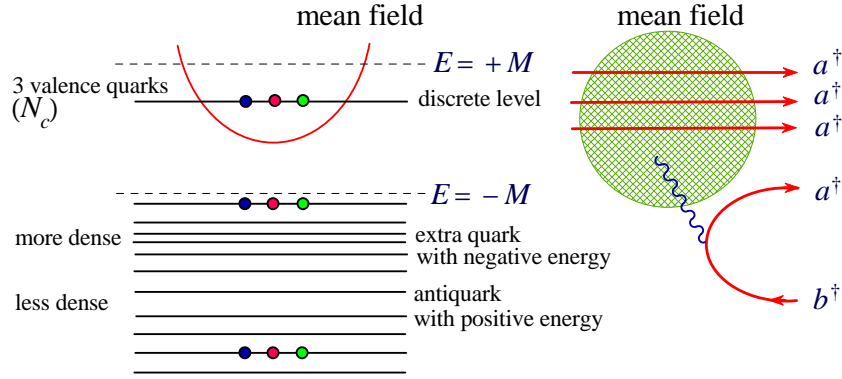


Fig. 1.2. *Left:* If the trial pion field is large enough (shown schematically by the solid curve), there is a discrete bound-state level for N_c ‘valence’ quarks, E_{val} . One has also to fill in the negative-energy Dirac sea of quarks (in the absence of the trial pion field it corresponds to the vacuum). The spectrum of the negative-energy levels is shifted in the trial pion field, its aggregate energy, as compared to the free case, being E_{sea} . *Right:* Equivalent view of baryons, where the polarized Dirac sea is presented as $Q\bar{Q}$ pairs.

to be very broad, the valence level dives towards the negative-energy Dirac sea, and there are $\mathcal{O}(N_c)$ (that is many!) additional $Q\bar{Q}$ pairs in a baryon, whose energy can be approximated by the two- and four-derivative Skyrme Lagrangian. The realistic picture is somewhere in between the two extremes.

Decoding the Skyrme model in terms of quarks allows one to answer important questions that cannot even be asked in the Skyrme model. For example, one can find out parton distributions in nucleons, satisfying all general sum rules and positivity constraints,¹⁷ the light-cone distribution amplitudes¹⁸ or, *e.g.* the wave functions of the 5-quark components in nucleons.¹⁶ For reviews of the model see.^{19,20}

To summarize this introduction: The original Skyrme’s idea is well founded from the modern QCD viewpoint. There is no mystics in the identification of the pion field winding number with the baryon number, and in the Skyrmin being a fermion (at odd N_c). The chiral soliton field, the Skyrmin, is nothing but the self-consistent mean field binding N_c valence quarks and distorting the Dirac sea such that additional $Q\bar{Q}$ pairs are necessary present in a baryon.

At the same time, one cannot expect a fully quantitative description of reality in the concrete two-terms Skyrme’s original model as an infinite series in the derivatives in the $E\chi\text{L}$ is truncated: it is similar to replacing e^{-x} by $1-x$. What is even worse, there are no explicit valence quarks in the Skyrme model as they cannot be separated from the sea.

In what follows, we shall nevertheless mainly deal with the concrete model by Skyrme (supplemented by the Wess–Zumino term) in order to study certain qualitative features of the exotic baryon resonances, *i.e.* those that by quantum numbers cannot be composed of three quarks only but need additional $Q\bar{Q}$ pairs.

1.2. Rotational states of the $SU(3)$ Skyrmion

The results of this section are general in the sense that they are independent on whether one takes literally the Skyrme model or a more sophisticated chiral quark model.

The standard choice of the saddle point field is the “upper-left corner hedgehog” Ansatz:

$$U_0(\mathbf{x}) \equiv e^{i\pi_0^A(\mathbf{x})\lambda^A} = \begin{pmatrix} e^{i(\mathbf{n}\cdot\boldsymbol{\tau})P(r)} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{n} = \frac{\mathbf{x}}{r}, \quad (1.3)$$

where λ^A are eight Gell-Mann matrices, τ^i are three Pauli matrices, and $P(r)$ is a spherically symmetric function called the profile of the Skyrmion. In the chiral limit $m_u = m_d = m_s = 0$ any $SU(3)$ rotation of the saddle point field, RU_0R^\dagger , $R \in SU(3)$, is also a saddle point. We consider a slowly rotating Ansatz,

$$U(\mathbf{x}, t) = R(t)U_0(\mathbf{x})R^\dagger(t) \quad (1.4)$$

and plug it into the E χ L. The degeneracy of the saddle point in the flavour rotations means that the action will not depend on R itself but only on the time derivatives \dot{R} . We do not consider the rotation angles as small but rather expand the action in angular velocities. In fact, one has to distinguish between the ‘right’ (Ω_A) and ‘left’ (ω_A) angular velocities defined as

$$\Omega_A = -i\text{Tr}(R^\dagger \dot{R} \lambda^A), \quad \omega_A = -i\text{Tr}(\dot{R} R^\dagger \lambda^A), \quad \Omega^2 = \omega^2 = 2\text{Tr} \dot{R}^\dagger \dot{R}. \quad (1.5)$$

Given the Ansatz (1.3) one expects on symmetry grounds the following Lagrangian for slow rotations:

$$\mathcal{L}_{\text{rot}} = \frac{I_1}{2} (\Omega_1^2 + \Omega_2^2 + \Omega_3^2) + \frac{I_2}{2} (\Omega_4^2 + \Omega_5^2 + \Omega_6^2 + \Omega_7^2) - \frac{N_c B}{2\sqrt{3}} \Omega_8 \quad (1.6)$$

where $I_{1,2}$ are the two soliton moments of inertia that are functionals of the profile function $P(r)$. Rotation along the 8th axis in flavour space, $R = \exp(i\alpha_8 \lambda^8)$, commutes with the ‘upper-left-corner’ Ansatz, therefore there is no quadratic term in Ω_8 . However there is a Wess–Zumino term resulting in a term linear in Ω_8 proportional to the baryon number B . In the chiral quark models this term arises from the extra bound-state levels for quarks.²¹

To quantize this rotational Lagrangian one uses the canonical quantization procedure. Namely, one introduces eight ‘right’ angular momenta J_A canonically conjugate to ‘right’ angular velocities Ω_A ,

$$J_A = -\frac{\partial \mathcal{L}_{\text{rot}}}{\partial \Omega_A}, \quad (1.7)$$

and writes the rotational Hamiltonian as

$$\mathcal{H}_{\text{rot}} = \Omega_A J_A - \mathcal{L}_{\text{rot}} = \frac{J_1^2 + J_2^2 + J_3^2}{2I_1} + \frac{J_4^2 + J_5^2 + J_6^2 + J_7^2}{2I_2} \quad (1.8)$$

with the additional quantization prescription following from Eq. (1.7),

$$J_8 = \frac{N_c B}{2\sqrt{3}}. \quad (1.9)$$

The quantization amounts to replacing classical angular momenta J_A by $SU(3)$ generators satisfying the $su(3)$ algebra: $[J_A J_B] = i f_{ABC} J_C$ where f_{ABC} are the $su(3)$ structure constants. These generators act on the matrix R on the right, $\exp(i\alpha^A J_A) R \exp(i(-\alpha_A J_A)) = R \exp(-i\alpha^A \lambda^A/2)$. For the first three generators ($A = 1, 2, 3$) this is equivalent, thanks to the hedgehog Ansatz (1.3), to rotating the space axes x, y, z . Therefore, $J_{1,2,3}$ are in fact spin generators.

One can also introduce ‘left’ angular momenta T_A canonically conjugate to the ‘left’ angular velocities ω_A ; they satisfy the same $su(3)$ algebra, $[T_A T_B] = i f_{ABC} T_C$, whereas $[T_A J_B] = 0$. These generators act on the matrix R on the left, $\exp(i\alpha^A T_A) R \exp(i(-\alpha_A T_A)) = \exp(i\alpha^A \lambda^A/2) R$, and hence have the meaning of $SU(3)$ flavour generators. The quadratic Casimir operator can be written using either ‘left’ or ‘right’ generators as

$$J_A J_A = T_A T_A = C_2(p, q) = \frac{1}{3} [p^2 + q^2 + pq + 3(p + q)] \quad (1.10)$$

where $C_2(p, q)$ is the eigenvalue of the quadratic Casimir operator for an irreducible representation r of $SU(3)$, labeled by two integers (p, q) . The rotational wave functions of chiral soliton are thus finite $SU(3)$ rotation matrices $D_{T, T_3, Y; J, J_3, Y'}^r(R)$ characterized by the eigenvalues of the commuting generators. For the $SU(2)$ group they are called Wigner finite-rotation matrices and depend on 3 Euler angles; in $SU(3)$ there are 8 ‘Euler’ angles. The general rotational functions (with important sign subtleties) are given in the Appendix of Ref.,²¹ and practically useful examples are given explicitly in Ref.¹⁶

One can visualize the rotational wave functions as a product of two same $SU(3)$ weight diagrams: one for the eigenvalues of the flavour (‘left’) generators, and the other for the eigenvalues of ‘right’ generators including the spin. Important, the quantization condition (1.9) means that not all $SU(3)$ representations can be viewed as rotational states of a Skyrmion. Taking baryon number $B = 1$ and $N_c = 3$ and recalling that $J_8 = Y' \sqrt{3}/2$ where Y is the hypercharge, the condition (1.9) means that only those multiplets are rotational states that contain particles with $Y' = 1$ or, more generally,

$$Y' = \frac{N_c}{3}. \quad (1.11)$$

The lowest $SU(3)$ multiplets meeting this condition are the octet, the decuplet and the antidecuplet, see Fig. 3.

It is remarkable that the lowest rotational states of the Skyrmion are exactly those observed in nature. The third is the antidecuplet with spin $\frac{1}{2}$. In the three vertices of the big triangle shown in Fig. 3, bottom right, there are baryons that are explicitly exotic, in the sense that they cannot be composed of three quarks but

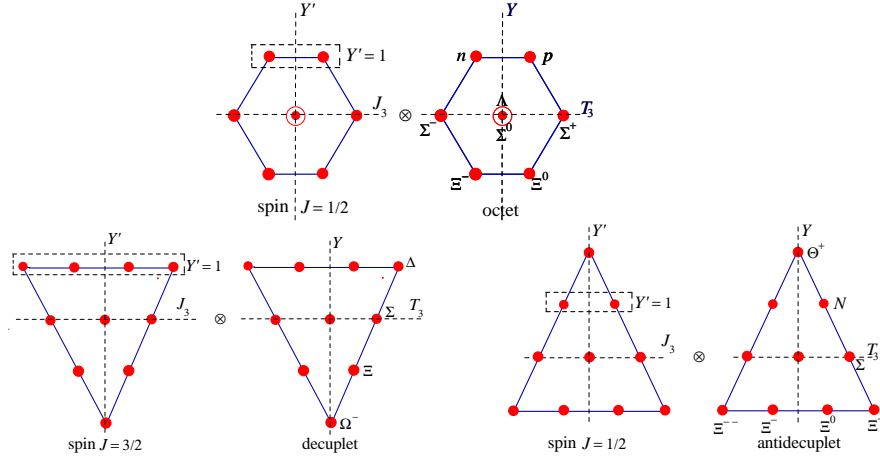


Fig. 1.3. The lowest rotational states of a Skyrmion, satisfying the condition $Y' = 1$: $(8, \frac{1}{2})$, $(10, \frac{3}{2})$, $(\overline{10}, \frac{1}{2})$. The number of states with $Y' = 1$, if one equates it to $2J + 1$, determines the spin J of the particles in the multiplet.

need an additional quark-antiquark pair. For example, the Θ^+ baryon at the top of the triangle can be composed minimally of $uudd\bar{s}$ quarks, *i.e.* it is a *pentaquark*. Seven baryons that are not in the vertices of the antidecuplet are cryptoexotic, in the sense that their quantum numbers can be, in principle, arranged from three quarks, however their expected properties are quite different from those of the similar members of a baryon octet.

It should be remembered, however, that strictly speaking the whole Skyrmion approach to baryons is justified in the limit of large N_c . Whether $N_c = 3$ is “large enough” is a question to which there is no unique answer: it depends on how large are the $1/N_c$ corrections to a particular physical quantity. Therefore, one has to be able to write equations with N_c being a free parameter. In particular, at arbitrary N_c one has to construct explicitly $SU(3)$ flavor multiplets that generalize the lightest baryon multiplets $(8, \frac{1}{2})$, $(10, \frac{3}{2})$, $(\overline{10}, \frac{1}{2})$, *etc.*, to arbitrary N_c . We do it in the next section following Ref.²² that generalizes previous work on this subject.^{23,24}

1.3. Rotational multiplets at arbitrary N_c

We remind the reader that a generic $SU(3)$ multiplet or irreducible representation is uniquely determined by two non-negative integers (p, q) having the meaning of upper (lower) components of the irreducible $SU(3)$ tensor $T_{\{g_1 \dots g_q\}}^{\{f_1 \dots f_p\}}$ symmetrized both in upper and lower indices and with a contraction with any $\delta_{f_m}^{g_m}$ being zero. Schematically, q is the number of boxes in the lower line of the Young tableau depicting an $SU(3)$ representation and p is the number of extra boxes in its upper line, see Fig. 4.

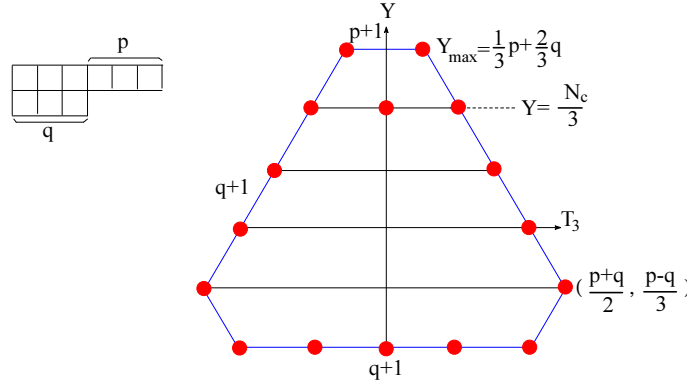


Fig. 1.4. A generic $SU(3)$ multiplet is, on the one hand, defined by the Young tableau and on the other hand can be characterized by quantum numbers (T_3, Y) of its members filling a hexagon in the (T_3, Y) axes (the weight diagram).

The dimension of a representation or the number of particles in the multiplet is

$$\text{Dim}(p, q) = (p+1)(q+1) \left(1 + \frac{p+q}{2} \right). \quad (1.12)$$

On the weight (T_3, Y) diagram where T_3 is the third projection of the isospin and Y is the hypercharge, a generic $SU(3)$ representation is depicted by a hexagon, whose upper horizontal side contains $p+1$ ‘dots’ or particles, the adjacent sides contain $q+1$ particles, with alternating $p+1$ and $q+1$ particles in the rest sides, the corners included – see Fig. 4. If either p or q is zero, the hexagon reduces to a triangle. Particles on the upper (horizontal) side of the hexagon have the hypercharge

$$Y_{\max} = \frac{1}{3}p + \frac{2}{3}q \quad (1.13)$$

being the maximal possible hypercharge of a multiplet with given (p, q) . Given that

$$\sum_{A=1}^3 J_A^2 = J(J+1), \quad \sum_{A=1}^8 J_A^2 = C_2(p, q), \quad J_8^2 = \frac{N_c^2}{12}, \quad (1.14)$$

one gets from Eq. (1.8) the rotational energy of baryons with given spin J and belonging to representation (p, q) :

$$\mathcal{E}_{\text{rot}}(p, q, J) = \frac{C_2(p, q) - J(J+1) - \frac{N_c^2}{12}}{2I_2} + \frac{J(J+1)}{2I_1}. \quad (1.15)$$

Only those multiplets are realized as rotational excitations that have members with hypercharge $Y = \frac{N_c}{3}$; if the number of particles with this hypercharge is n the spin J of the multiplet is such that $2J+1 = n$. It is easily seen that the number of particles with a given Y is $\frac{4}{3}p + \frac{2}{3}q + 1 - Y$ and hence the spin of the allowed multiplet is

$$J = \frac{1}{6}(4p + 2q - N_c). \quad (1.16)$$

A common mass \mathcal{M}_0 must be added to Eq. (1.15) to get the mass of a particular multiplet. Throughout this section we disregard the splittings inside multiplets as due to nonzero current strange quark mass.

The condition that a horizontal line $Y = \frac{N_c}{3}$ must be inside the weight diagram for the allowed multiplet leads to the requirement

$$\frac{N_c}{3} \leq Y_{\max} \quad \text{or} \quad p + 2q \geq N_c \quad (1.17)$$

showing that at large N_c multiplets must have a high dimension!

We introduce a non-negative number which we name “exoticness” X of a multiplet defined as²²

$$Y_{\max} = \frac{1}{3}p + \frac{2}{3}q \equiv \frac{N_c}{3} + X, \quad X \geq 0. \quad (1.18)$$

Combining Eqs.(1.16,1.18) we express (p, q) through (J, X) :

$$\begin{aligned} p &= 2J - X, \\ q &= \frac{1}{2}N_c + 2X - J. \end{aligned} \quad (1.19)$$

The total number of boxes in Young tableau is $2q + p = N_c + 3X$. Since we are dealing with unity baryon number states, the number of quarks in the multiplets we discuss is N_c , *plus* some number of quark-antiquark pairs. In the Young tableau, quarks are presented by single boxes and antiquarks by double boxes. It explains the name “exoticness”: X gives the minimal number of additional quark-antiquark pairs one needs to add on top of the usual N_c quarks to compose a multiplet.

Putting (p, q) from Eq. (1.19) into Eq. (1.15) we obtain the rotational energy of a soliton as function of the spin and exoticness of the multiplet:

$$\mathcal{E}_{\text{rot}}(J, X) = \frac{X^2 + X(\frac{N_c}{2} + 1 - J) + \frac{N_c}{2}}{2I_2} + \frac{J(J+1)}{2I_1}. \quad (1.20)$$

We see that for given $J \leq \frac{N_c}{2} + 1$ the multiplet mass is a monotonically growing function of X : the minimal-mass multiplet has $X = 0$. Masses of multiplets with increasing exoticness are:

$$\mathcal{M}_{X=0}(J) = \mathcal{M}'_0 + \frac{J(J+1)}{2I_1}, \quad \text{where} \quad \mathcal{M}'_0 \equiv \mathcal{M}_0 + \frac{N_c}{4I_2}, \quad (1.21)$$

$$\mathcal{M}_{X=1}(J) = \mathcal{M}'_0 + \frac{J(J+1)}{2I_1} + 1 \cdot \frac{\frac{N_c}{2} + 2 - J}{2I_2}, \quad (1.22)$$

$$\mathcal{M}_{X=2}(J) = \mathcal{M}'_0 + \frac{J(J+1)}{2I_1} + 2 \cdot \frac{\frac{N_c}{2} + 2 - J}{2I_2} + \frac{1}{I_2}, \quad \text{etc.} \quad (1.23)$$

At this point it should be recalled that both moments of inertia $I_{1,2} = O(N_c)$, as is \mathcal{M}_0 . We see from Eqs.(1.21)-(1.23) that multiplets fall into a sequence of rotational bands each labeled by its exoticness with small $O(1/N_c)$ splittings inside

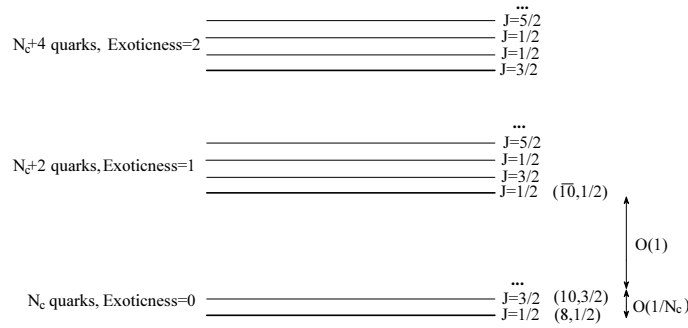


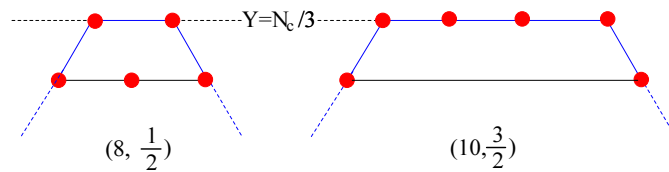
Fig. 1.5. Rotational excitations form a sequence of bands.

the bands. The separation between bands with different exoticness is $O(1)$. The corresponding masses are schematically shown in Fig. 5.

The lowest band is non-exotic ($X=0$); the multiplets are determined by $(p, q) = (2J, \frac{N_c}{2} - J)$, and their dimension is $\text{Dim} = (2J+1)(N_c+2-2J)(N_c+4+2J)/8$ which in the particular (but interesting) case of $N_c = 3$ becomes 8 for spin one half and 10 for spin 3/2. These are the correct lowest multiplets in real world, and the above multiplets are their generalization to arbitrary values of N_c . To make baryons fermions one needs to consider only odd N_c .

Recalling that u, d, s quarks' hypercharges are $1/3, 1/3$ and $-2/3$, respectively, one observes that all baryons of the non-exotic $X=0$ band can be made of N_c quarks. The upper side of their weight diagrams (see Fig. 6) is composed of u, d quarks only; in the lower lines one consequently replaces u, d quarks by the s one. This is how the real-world $(8, \frac{1}{2})$ and $(10, \frac{3}{2})$ multiplets are arranged and this property is preserved in their higher- N_c generalizations. The construction coincides with that of Ref.²³ At high N_c there are further multiplets with spin 5/2 and so on. The maximal possible spin at given N_c is $J_{\max} = \frac{N_c}{2}$: if one attempts higher spin, q becomes negative.

The rotational bands for $X=1$ multiplets are shown in Fig. 7, left and middle graphs. The upper side of the weight diagram is exactly one unit higher than the line $Y = \frac{N_c}{3}$ which is non-exotic, in the sense that its quantum numbers can be, in principle, achieved from exactly N_c quarks. However, particles corresponding to

Fig. 1.6. Non-exotic ($X=0$) multiplets that can be composed of N_c quarks.

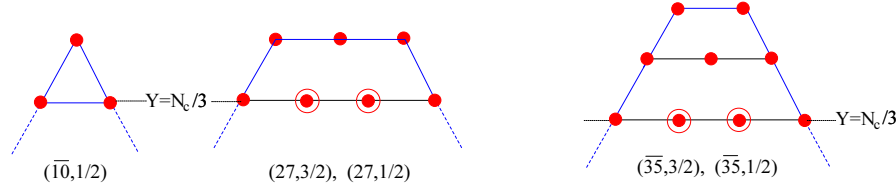


Fig. 1.7. Exotic ($X=1$) multiplets (left and middle graphs) that can be composed of N_c quarks and one extra $Q\bar{Q}$ pair. An example of an $X=2$ multiplet that can be composed with two additional $Q\bar{Q}$ pairs, is shown on the right.

the upper side of the weight diagram cannot be composed of N_c quarks but require at least one additional \bar{s} quark and hence *one additional quark-antiquark pair* on top of N_c quarks.

The multiplet shown in Fig. 7, left, has only one particle with $Y = \frac{N_c}{3} + 1$. It is an isosinglet with spin $J = \frac{1}{2}$, and in the quark language is built of $(N_c + 1)/2$ ud pairs and one \bar{s} quark. It is the generalization of the Θ^+ baryon to arbitrary odd N_c . As seen from Eqs.(1.12,1.19), the multiplet to which the “ Θ^+ ” belongs is characterized by $(p, q) = (0, (N_c + 3)/2)$, its dimension is $(N_c + 5)(N_c + 7)/8$ becoming the $(\overline{10}, \frac{1}{2})$ at $N_c = 3$. Its splitting with the N_c generalization of the non-exotic $(8, \frac{1}{2})$ multiplet follows from Eq. (1.22):

$$\mathcal{M}_{\overline{10}, \frac{1}{2}} - \mathcal{M}_{8, \frac{1}{2}} = \frac{N_c + 3}{4I_2}, \quad (1.24)$$

a result first found in Ref.²⁴ Here and in what follows we denote baryon multiplets by their dimension at $N_c=3$ although at $N_c>3$ their dimension is higher, as given by Eq. (1.12).

The second rotational state of the $X=1$ sequence has $J = \frac{3}{2}$; it has $(p, q) = (2, (N_c + 1)/2)$ and dimension $3(N_c + 3)(N_c + 9)/8$ reducing to the multiplet $(27, \frac{3}{2})$ at $N_c = 3$, see Fig. 7, middle. In fact there are two physically distinct multiplets there. Indeed, the weights in the middle of the second line from top on the weight diagram with $Y = \frac{N_c}{3}$ are twice degenerate, corresponding to spin 3/2 and 1/2. Therefore, there is another $3(N_c + 3)(N_c + 9)/8$ -plet with unit exoticness, but with spin 1/2. At $N_c=3$ it reduces to $(27, \frac{1}{2})$. The splittings with non-exotic multiplets are

$$\mathcal{M}_{27, \frac{3}{2}} - \mathcal{M}_{10, \frac{3}{2}} = \frac{N_c + 1}{4I_2}, \quad (1.25)$$

$$\mathcal{M}_{27, \frac{1}{2}} - \mathcal{M}_{8, \frac{1}{2}} = \frac{N_c + 7}{4I_2}. \quad (1.26)$$

The $X=1$ band continues to the maximal spin $J_{\max} = (N_c + 4)/2$ where q becomes zero.

The $X=2$ rotational band (see Fig. 7, right) starts from two states with spin 3/2 and 1/2 both belonging to the $SU(3)$ representation $(p, q, \text{Dim}) =$

$(1, (N_c + 5)/2, (N_c + 7)(N_c + 11)/4)$. It reduces to the $\overline{\mathbf{35}}$ multiplet at $N_c = 3$. Their splittings with non-exotic multiplets are

$$\mathcal{M}_{\overline{\mathbf{35}}, \frac{3}{2}} - \mathcal{M}_{10, \frac{3}{2}} = \frac{N_c + 3}{2I_2}, \quad (1.27)$$

$$\mathcal{M}_{\overline{\mathbf{35}}, \frac{1}{2}} - \mathcal{M}_{8, \frac{1}{2}} = \frac{N_c + 6}{2I_2}. \quad (1.28)$$

The maximal spin of the $X=2$ rotational band is $J_{\max} = (N_c + 8)/2$.

The upper side in the weight diagram in Fig. 7, right, for the $X=2$ sequence has hypercharge $Y_{\max} = \frac{N_c}{3} + 2$. Therefore, one needs *two* \bar{s} quarks to get that hypercharge and hence the multiplets can be minimally constructed of N_c quarks plus *two additional quark-antiquark pairs*.

Disregarding the rotation along the 1,2,3 axes (for example taking only the lowest J state from each band) we observe from Eq. (1.20) that at large N_c the spectrum is equidistant in exoticness,

$$\mathcal{E}_{\text{rot}}(X) = \frac{N_c(X+1)}{4I_2}, \quad (1.29)$$

with the spacing $\frac{N_c}{4I_2} = O(1)$. It is consistent with the fact explained in the next Section, that at large N_c the rotation corresponding to the excitations of exoticness is actually a small-angle precession equivalent to small oscillations whose quantization leads to an equidistant spectrum. We stress that there is no deformation of the Skyrmion by rotation until X becomes of the order of N_c .²² Eq. (1.29) means that each time we add a quark-antiquark pair it costs at large N_c the same

$$\text{energy of a } Q\bar{Q} \text{ pair} = \omega_{\text{rot}} = \frac{N_c}{4I_2} = O(N_c^0). \quad (1.30)$$

Naively one may think that this quantity should be approximately twice the constituent quark mass $M \approx 350$ MeV. Actually, it can be much less than that. For example, an inspection of I_2 in the Chiral Quark Soliton Model^{9,21} shows that the pair energy is strictly less than $2M$; in fact $1/I_2$ tends to zero in the limit when the baryon size blows up.

In physical terms, the energy cost of adding a $Q\bar{Q}$ pair can be small if the pair is added in the form of a Goldstone boson. The energy penalty for making, say, the Θ^+ baryon from a nucleon would be exactly zero in the chiral limit and were baryons infinitely large. In reality, one has to create a pseudo-Goldstone K-meson and to confine it inside the baryon of the size $\geq 1/M$. It costs roughly

$$m(\Theta) - m(N) \approx \sqrt{m_K^2 + \mathbf{p}^2} \leq \sqrt{495^2 + 350^2} = 606 \text{ MeV}. \quad (1.31)$$

Therefore, one should expect the exotic Θ^+ around 1540 MeV where indeed it has been detected in a number of experiments!

1.4. Rotational wave functions

It is helpful to realize how do the rotational wave functions $\Psi(R)$ look like for various known (and unknown) baryons. To that end, one needs a concrete parameterization of the $SU(3)$ rotation matrix R by 8 ‘Euler’ angles: the wave functions are in fact functions of those angles.

In general, the parameter space of an $SU(N)$ group is a direct product of odd-dimensional spheres, $S^3 \times S^5 \times \dots \times S^{2N-1}$. For $SU(3)$, it is a product of the spheres $S^3 \times S^5$. A general $SU(3)$ matrix R can be written as $R = S_3 R_2$ where R_2 is a general $SU(2)$ matrix with three parameters, put in the upper-left corner, and S_3 is an $SU(3)$ matrix of a special type with five parameters, see Appendix A in Ref.¹⁶

To be specific, let us consider the rotational wave function corresponding to the exotic Θ^+ baryon. For general N_c its (complex conjugate) wave function is given by¹⁶

$$\Theta_k(R)^* = (R_3^3)^{N_c-1} R_k^3 \quad (1.32)$$

where $k = 1, 2$ is the spin projection and R_k^3 is the k^{th} matrix element in the 3^{d} row of the 3×3 matrix R . Using the concrete parameterization of Ref.¹⁶ Eq. (1.32) becomes

$$\Theta(R)^* \sim (\cos \theta \cos \phi)^{N_c}, \quad (1.33)$$

where $\theta, \phi \in (0, \frac{\pi}{2})$ are certain angles parameterizing the S^5 sphere; $\theta = \phi = 0$ corresponds to the North pole of that sphere. We see that although for $N_c=3$ the typical angles in the wave function are large such that it is spread over both S^3 and S^5 globes, at $N_c \rightarrow \infty$ the wave function is concentrated near the North pole of S^5 since

$$\theta \sim \phi \sim \sqrt{\frac{2}{N_c}} \xrightarrow{N_c \rightarrow \infty} 0. \quad (1.34)$$

This is illustrated in Fig. 8.

Let us show that the limit $N_c \rightarrow \infty$ corresponds to the weak kaon field in the Θ^+ baryon. To that end we use an alternative parameterization for the meson field fluctuations about the Skyrmion, suggested by Callan and Klebanov:²⁵

$$U = \sqrt{U_0} U_K \sqrt{U_0}, \quad (1.35)$$

where U_K is the meson $SU(3)$ unitary matrix which, for small meson fluctuations ϕ^A about the saddle-point Skyrmion field U_0 (1.3), is

$$U_K = \mathbf{1}_3 + i\phi^A \lambda^A, \quad A = 1 \dots 8, \quad (1.36)$$

$$\pi^\pm = \frac{\phi^1 \pm i\phi^2}{\sqrt{2}}, \quad \pi^0 = \phi^3, \quad K^\pm = \frac{\phi^4 \pm i\phi^5}{\sqrt{2}}, \quad K^0, \bar{K}^0 = \frac{\phi^6 \pm i\phi^7}{\sqrt{2}}, \quad \eta = \phi^8.$$

One can compare Eq. (1.35) with the rotational Ansatz, $U = R U_0 R^\dagger$, and find the meson fields in baryons corresponding to rotations. In particular, for rotations

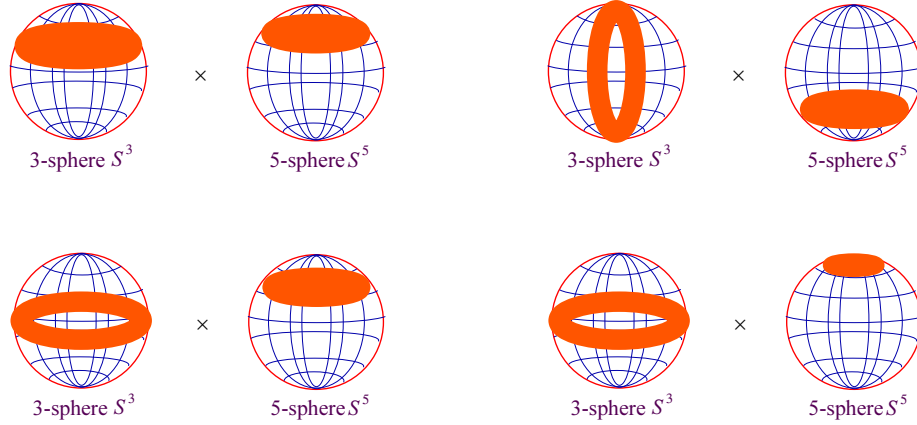


Fig. 1.8. A schematic view of the rotational wave functions of several baryons. The shaded areas indicate where the wave functions of the eight ‘Euler’ angles that parameterize the S^3, S^5 spheres, are large. Top left: proton, spin up; top right: Ω^- , spin down-down; bottom left: Θ^+ , spin up; bottom, right: Θ^+ , spin up at $N_c = 37$.

“near the North pole” *i.e.* at small angles θ, ϕ , one finds the kaon field¹⁶

$$\left. \begin{aligned} K^+ &= -\sqrt{2} \sin \frac{P(r)}{2} [\theta n_z + \phi (n_x - i n_y)] \\ K^0 &= -\sqrt{2} \sin \frac{P(r)}{2} [\theta (n_x + i n_y) - \phi n_z] \end{aligned} \right\} = -\sqrt{2} \sin \frac{P(r)}{2} (\mathbf{n} \cdot \boldsymbol{\tau}) \begin{pmatrix} \theta \\ \phi \end{pmatrix}, \quad (1.37)$$

meaning that at large N_c the amplitude of the kaon fluctuations in the prototype “ Θ ” is vanishing as $\sim 1/\sqrt{N_c}$. Therefore, at large N_c the rotation is in fact a small-angle precession about the North pole, that can be studied as a small kaon field fluctuation about the Skyrmion in a given particular model for the $E\chi L$.^{26,27} It should be kept in mind, however, that in reality at $N_c=3$ the rotations by large angles θ, ϕ are not suppressed. It means that in the real world the kaon field in the Θ^+ is generally not small.

1.5. Kaons scattering off the Skyrmion

As explained in the previous section, at large N_c the kaon field in the exotic baryon Θ^+ is weak, hence the resonance should manifest itself in the linear order in the kaon field perturbing the nucleon which, again at large N_c , can be represented by a Skyrmion.^{25,27} In this section we look for the Θ^+ by studying small kaon field fluctuations about the Skyrmion taking as a model for the $E\chi L$ the Skyrme

Lagrangian:

$$S = S_{\text{kin}} + S_{\text{Sk}} + S_{\text{WZ}} + S_m, \quad (1.38)$$

$$S_{\text{kin}} = \frac{F_\pi^2}{4} \int d^4x \text{Tr} L_\mu L_\mu, \quad L_\mu := iU^\dagger \partial_\mu U, \quad (1.39)$$

$$S_{\text{Sk}} = -\frac{1}{32e^2} \int d^4x \text{Tr} [L_\mu L_\nu]^2, \quad (1.40)$$

$$S_{\text{WZ}} = \frac{N_c}{24\pi^2} \int d^4x \int_0^1 ds \epsilon^{\alpha\beta\gamma\delta} \text{Tr} \left(e^{-is\Pi} \partial_\alpha e^{is\Pi} \right) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}_\beta \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}_\gamma \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}_\delta, \quad e^{i\Pi} = U, \quad (1.41)$$

$$S_m = \int d^4x \frac{m_K^2 F_\pi^2}{2} \text{Tr} \left[(U + U^\dagger - 2 \cdot \mathbf{1}_3) \text{diag} \left(\frac{m_u}{m_s}, \frac{m_d}{m_s}, 1 \right) \right]. \quad (1.42)$$

We have written the Wess–Zumino term (1.41) in the explicit form suggested in Ref.⁶ In the last, symmetry breaking term, we shall put $m_{u,d} = 0$.

Following the general approach of Callan and Klebanov²⁵ revived by Klebanov *et al.*²⁷ in the pentaquark era, we use the parameterization of $U(\mathbf{x}, t)$ in the form of Eq. (1.36) where we take the small kaon fluctuation in the form hinted by Eq. (1.37):

$$K^\alpha(\mathbf{x}, t) = (\mathbf{n} \cdot \boldsymbol{\tau})_\beta^\alpha \zeta^\beta \eta(r) e^{-i\omega t} \quad (1.43)$$

where ζ^β is a constant spinor. It corresponds to the p -wave kaon field.

Expanding the action (1.38) in the kaon field up to the second order one obtains^{25,27} (we measure r, t in conventional units of $1/(2F_\pi e) = \mathcal{O}(N_c^0)$)

$$S = S_0 + S_2, \quad (1.44)$$

$$S_0 = \frac{2\pi F_\pi}{e} \int dr r^2 \left[\frac{d(r)}{2} (1 + 2s(r)) + s(r) \left(1 + \frac{s(r)}{2} \right) \right], \quad (1.45)$$

$$S_2 = \frac{4\pi F_\pi}{e} \zeta^\dagger \zeta \int dr r^2 \eta(r) \left\{ \omega^2 A(r) - 2\omega \gamma B(r) + \left[C(r) \frac{d^2}{dr^2} + D(r) \frac{d}{dr} - V(r) \right] \right\} \eta(r) \quad (1.46)$$

where one introduces short-hand notations:

$$b(r) := P''(r) \sin P(r) + P'^2(r) \cos P(r), \quad c(r) := \sin^2 \frac{P(r)}{2}, \quad d(r) := P'^2(r),$$

$$h(r) := \sin(2P(r)) P'(r), \quad s(r) := \frac{\sin^2 P(r)}{r^2},$$

$$A(r) := 1 + 2s(r) + d(r), \quad B(r) := -N_c e^2 \frac{P'(r) \sin^2 P(r)}{2\pi^2 r^2},$$

$$C(r) := 1 + 2s(r), \quad D(r) := \frac{2}{r} [1 + h(r)],$$

$$V(r) := -\frac{1}{4} [d(r) + 2s(r)] - 2s(r) [s(r) + 2d(r)] + 2 \frac{1 + d(r) + s(r)}{r^2} [1 - c(r)]^2 + \frac{6}{r^2} [s(r) (1 - c(r))^2 - b(r)(1 - c(r)) + \frac{1}{2} r^2 d(r) s(r)] + \mu_K^2. \quad (1.47)$$

Here μ_K is the dimensionless kaon mass, $\mu_K = m_K/(2F_\pi e)$. The term linear in ω in Eq. (1.46) arises from the Wess–Zumino term (1.41); the function $B(r)$ is the baryons number density in the Skyrme model. The coefficient γ in front of it is unity in the chiral limit but in general is not universal. In what follows it is useful to analyze the results as one varies γ from 0 to 1.

Varying S_0 with respect to $P(r)$ one finds the standard Skyrmion profile with $P(0) = \pi$ and $P(r) \xrightarrow{r \rightarrow \infty} r_0^2/r^2$. Varying S_2 with respect to the kaon field profile $\eta(r)$ one obtains a Schrödinger-type equation

$$\left\{ \omega^2 A(r) - 2\omega\gamma B(r) + \left[C(r) \frac{d^2}{dr^2} + D(r) \frac{d}{dr} - V(r) \right] \right\} \eta(r) = 0 \quad (1.48)$$

where the profile $P(r)$ found from the minimization of S_0 has to be substituted. In the chiral limit ($m_K \rightarrow 0$) the equations for $P(r)$, $\eta(r)$ are equivalent to the conservation of the axial current, $\partial_\mu j_{\mu 5}^A = 0$ since it is the equation of motion for the Skyrme model.

If $m_K = 0$, the $SU(3)$ symmetry is exact, and a small and slow rotation in the strange direction must be a zero mode of Eq. (1.48). Indeed, one can easily check that

$$\eta_{\text{rot}}(r) = \sin \frac{P(r)}{2} \quad (1.49)$$

is a zero mode of the square brackets in Eq. (1.48) and hence a zero mode of the full equation with $\omega = 0$. If in addition the Wess–Zumino coefficient γ is set to zero, this mode is twice degenerate. These states are the large- N_c prototypes of Λ (strangeness $S = -1$) and Θ^+ ($S = +1$).^{25,27} At $\gamma > 0$ the two states split: Λ remains a pole of the scattering amplitude at $\omega = 0$, and Θ^+ moves into the lower semi-plane of the complex ω plane. If $m_K > 0$ the pole corresponding to the Λ moves to $\omega < 0$ remaining on the real axis, whereas the Θ^+ pole remains in the lower semi-plane with $\text{Re } \omega > 0$ and $\text{Im } \omega < 0$. Both poles are singularities of the same analytical function *i.e.* the scattering amplitude, see below. It is amusing that Λ “knows” about Θ^+ and its width through analyticity.

In what follows we shall carefully study the solutions of Eq. (1.48) and in particular the trajectory of the Θ^+ pole, by combining numerical and analytical calculations. In numerics, we use the conventional choice of the constants in the Skyrme model: $F_\pi = 64.5$ MeV (*vs* 93 MeV experimentally) and $e = 5.45$. These values fit the nucleon mass $m_N = 940$ MeV (with the account for its rotational energy) and the mass splitting between the nucleon and the Δ -resonance.²⁸ These were the values used also by Klebanov *et al.*²⁷ who solved numerically Eq. (1.48) and found the phase shifts $\delta(\omega)$ defined from the large- r asymptotics of the solutions of Eq. (1.48) regular at the origin,

$$\eta_{\text{as}}(r) = \frac{kr + i}{r^2} e^{ikr + i\delta(\omega)} + \frac{kr - i}{r^2} e^{-ikr - i\delta(\omega)}, \quad (k = \sqrt{\omega^2 - m_K^2}), \quad (1.50)$$

being a superposition of the incoming and outgoing spherical waves. At $\gamma = 1$ and physical $m_K = 495$ MeV, the phase shift $\delta(\omega)$ has been found in Ref.²⁷ to be less

than 45° in the range of interest. This have lead the authors to the conclusion that Θ^+ does not exist in the Skyrme model, at least in the large N_c limit and small m_K . We reproduce their phase shifts with a high accuracy (as well as the phase shifts studied in Ref.²⁹ for another choice of the Skyrme model parameters) but come to the opposite conclusions.

In a situation when there is a resonance and a potential scattering together, the phase shift does not need to go through 90° as it would be requested by the Breit–Wigner formula for an isolated resonance. A far better and precise way to determine whether there is a resonance, is to look not into the phase shifts but into the singularities of the scattering amplitude in the complex energy plane. A resonance is, by definition, a pole of the scattering amplitude in the lower semi-plane on the second Riemann sheet:

$$\sqrt{s}_{\text{pole}} = m_{\text{res}} - i\frac{\Gamma}{2} \quad (1.51)$$

where m_{res} is the resonance mass and Γ is its width.

The scattering amplitude $f(\omega)$ and the scattering matrix $S(\omega)$ (which in this case has only one element) are defined as

$$f(\omega) = \frac{1}{2ik} (e^{2i\delta(\omega)} - 1), \quad S(\omega) = e^{2i\delta(\omega)}. \quad (1.52)$$

A standard representation for the scattering amplitude is

$$f = \frac{1}{g(\omega) - ik}, \quad g(\omega) = k \cot \delta(\omega). \quad (1.53)$$

This representation solves the unitarity condition for the S -matrix: g is real on the real ω axis. The function $g(\omega)$ does not have cuts related to the KN thresholds and $\omega^{2l}g(\omega)$ is Taylor-expandable at small ω , therefore it is a useful concept.³⁰

For ω in the lower complex semi-plane the first term in Eq. (1.50) becomes a rising exponent of r , and the second term becomes a falling exponent. Since the S matrix is proportional to the ratio of the coefficient in front of $\exp(-ikr)$ to that in front of $\exp(ikr)$, the pole of the S matrix and hence of the scattering amplitude corresponds to the situation where the wave function $\eta(r)$ regular at the origin, has no falling exponent at $r \rightarrow \infty$ but only a rising one. Physically, it corresponds to a resonance decay producing outgoing waves only.

For the conventional choice of the parameters we find the Θ^+ pole position at

$$\sqrt{s}_{\text{pole}} = \begin{cases} (1115 - 145i) \text{ MeV} & \text{for } m_K = 0 \quad (\text{threshold at } 940 \text{ MeV}) \\ (1449 - 44i) \text{ MeV} & \text{for } m_K = 495 \text{ MeV} \quad (\text{threshold at } 1435 \text{ MeV}) \end{cases} \quad (1.54)$$

We have recalculated here the pole position in ω to the relativistic-invariant KN energy $s = m_N^2 + 2m_N\omega + m_K^2$. It is a perfectly normal resonance in the strong interactions standards with a width of 90 MeV. It would be by all means seen in a partial wave analysis (see Fig. 9, left) or just in the KN total $T = 0$ cross

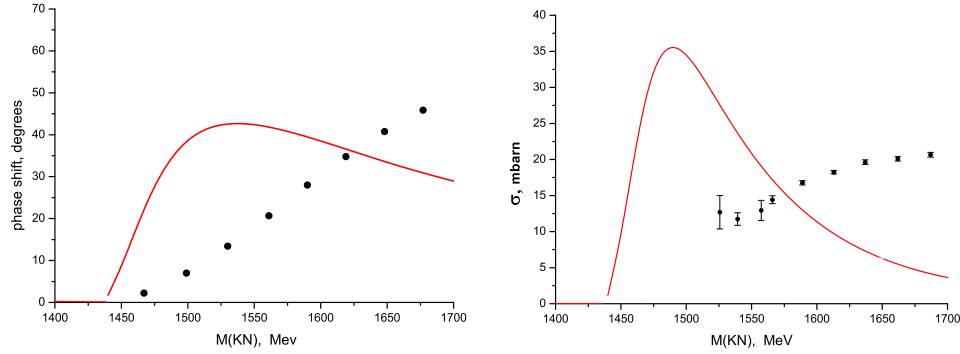


Fig. 1.9. *Left:* the $T=0, L=1$ KN scattering phase as function of the KN invariant mass in the Skyrme model in the large- N_c limit (it coincides with the phase found in Ref.²⁷), compared to the result of the partial wave analysis³¹ shown by dots. *Right:* the ensuing KN cross section in this partial wave exhibits a strong resonance around 1500 MeV, whereas the experimental data³² for the sum over all partial waves shows no signs of a resonance.

section which we calculate from the well-known equation $\sigma = 4\pi(2j+1)|f|^2 = \frac{4\pi}{k^2}(2j+1)\sin^2\delta$ (Fig. 9, right).

At the maximum the cross section is as large as 35 mb, and it is a contribution of only one particular partial wave P_{01} ! Needless to say, such a strong resonance is not observed. Varying the parameters of the Skyrme model^{27,29} or modifying it²⁶ can make the exotic resonance narrower or broader but one cannot get rid of it. The reason is very general: poles in the scattering amplitude do not disappear as one varies the parameters but move in the complex plane.

One can check it in a very precise way by, say, varying artificially the coefficient in front of the Wess–Zumino term γ from 0 to 1. At $\gamma = 0$ there is certainly an exotic bound state at $\omega = 0$ corresponding to the rotational zero mode (1.49). At $\gamma > 0$ the position of the pole of the KN scattering amplitude moves into the complex ω plane such that

$$\begin{aligned} \text{Re } \omega_{\text{pole}} &= a_1\gamma + a_3\gamma^3 + \dots, \\ \text{Im } \omega_{\text{pole}} &= b_2\gamma^2 + b_4\gamma^4 + \dots \end{aligned} \quad (1.55)$$

with analytically calculable coefficients in this Taylor expansion (we give explicitly the leading coefficients in Section 7). By comparing the numerical determination of the pole position with the analytical expressions we trace that the Θ^+ pole (1.54) is a continuous deformation of the rotational mode, see Fig. 10.

Thus, the prediction of the Skyrme model is not that there is no Θ^+ but just the opposite: **there must be a very strong resonance**, at least when the number of colours is taken to infinity. Since this prediction is of general nature and does not rely on the specifics of the Skyrme model, one must be worried why a strong exotic resonance is not observed experimentally!

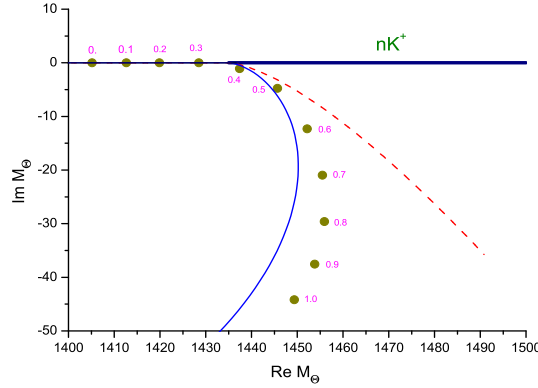


Fig. 1.10. Trajectory of the pole in the KN scattering amplitude for “realistic” parameters of the Skyrme model and physical $m_K = 495$ MeV at $\gamma = 0, 0.1, \dots, 1.0$. The dashed and solid lines show the analytical calculation of the pole position in the first and second orders in γ , respectively.

The answer is that the large- N_c logic in general and the concrete Skyrme model in particular grossly overestimate the resonance width (we explain it in the next Sections). The resonance cannot disappear but in reality it becomes very narrow, and that is why it is so difficult to observe it.

One may object that the Skyrme model is a model anyway, and a modification of its parameters or a replacement by another chiral model can lead to an even larger width, say, of 600 MeV instead of 90 MeV obtained here from the “classical” Skyrme model. However, as we argue in Section 9, going from $N_c = \infty$ to the real world at $N_c = 3$ reduces the width by at least a factor of 5. Therefore, even a 600-MeV resonance at $N_c = \infty$ would become a normal 120-MeV resonance in the real world and would be observable.

Thus, the only way how a theoretically unavoidable resonance can escape observation is to become very narrow. We remark that the reanalysis of the old KN scattering data³³ shows that there is room for the exotic resonance with a mass around 1530 MeV and width below 1 MeV.

1.6. Physics of the narrow Θ^+ width

Quantum field theory says that baryons cannot be $3Q$ states only but necessarily have higher Fock components due to additional $Q\bar{Q}$ pairs; it is only a quantitative question how large are the $5Q$, $7Q$, ... components in ordinary baryons. Various baryon observables have varying sensitivity to the presence of higher Fock components. For example, the fraction of the nucleon momentum carried by antiquarks is, at low virtuality, less than 10%. However, the nucleon σ -term or nucleon spin are in fact dominated by antiquarks.^{34,35} Both facts are in accord with a normalization

of the $5Q$ component of the nucleon at the level of 30% from the $3Q$ component, meaning that 30% of the time nucleon is a pentaquark!

As to the exotic Θ^+ and other members of the antidecuplet, their *lowest* Fock component is the $5Q$ one, nothing terrible. However it has dramatic consequences for the antidecuplet decay widths.

To evaluate the width of the $\Theta^+ \rightarrow K^+ n$ decay one has to compute the transition matrix element of the strange axial current, $\langle \Theta^+ | \bar{s} \gamma_\mu \gamma_5 u | n \rangle$. There are two contributions to this matrix element: the “fall apart” process (Fig. 10, A) and the “5-to-5” process where Θ^+ decays into the $5Q$ component of the nucleon (Fig. 10, B). One does not exist without the other: if there is a “fall apart” process it means that there is a non-zero coupling of quarks to pseudoscalar (and other) mesons, meaning that there is a transition term in the Hamiltonian between $3Q$ and $5Q$ states (Fig. 10, C). Hence the eigenstates of the Hamiltonian must be a mixture of $3Q, 5Q, \dots$ Fock components. Therefore, assuming there is process A, we have to admit that there is process B as well. Moreover, each of the amplitudes A and B are not Lorentz-invariant, only their sum is. Evaluating the “fall-apart” amplitude and forgetting about the “5-to-5” one makes no sense.

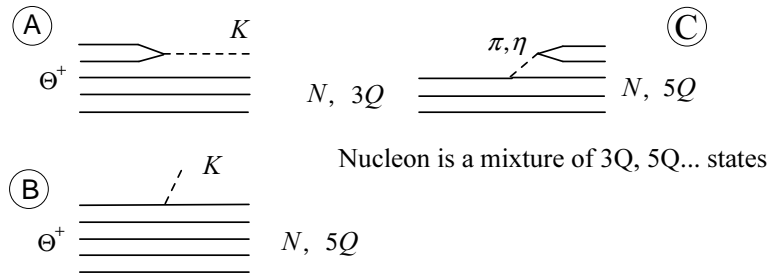


Fig. 1.11. “Fall-apart” (A) and “5-to-5” (B) contributions to the $\Theta^+ \rightarrow K^+ n$ decay.

A convenient way to evaluate the sum of two graphs, A and B, in the chiral limit is to go to the infinite momentum frame (IMF) where only the process B survives, as axial (and vector) currents with a finite momentum transfer do not create or annihilate quarks with infinite momenta. In the IMF the fall-apart process A is exactly zero in the chiral limit. The baryon matrix elements are thus non-zero only between Fock components with *equal number* of quarks and antiquarks.

The decays of ordinary (nonexotic) baryons are mainly due to the $3Q \rightarrow 3Q$ transitions with a small (30%) correction from $5Q \rightarrow 5Q$ transitions and even smaller corrections from higher Fock components, just because the $3Q$ components dominate. The nucleon axial constant is thus expected to be of the order of unity and indeed $g_A(N) = 1.27$.

However the Θ^+ decay is dominated by the $5Q \rightarrow 5Q$ transition, and **the transition axial constant is suppressed to the extent the $5Q$ component in the nucleon is suppressed with respect to the $3Q$ one.**³⁶

A quantitative estimate of this effect can be made in a relativistic model (since pair creation and annihilation is a relativistic effect) where it is possible to calculate both the $3Q$ and the $5Q$ wave functions of the nucleon and the $5Q$ wave function of the Θ^+ . We claim that in any such model of baryons the Θ^+ will be narrow if the model tells that the $5Q$ component in the nucleon is suppressed with respect to the $3Q$ one. An example is provided by the Chiral Quark Soliton Model where, indeed, the $5Q \rightarrow 5Q$ axial constant has been estimated as $g_A(\Theta \rightarrow KN) \approx 0.14$ ^{16,37} yielding

$$\Gamma_\Theta \approx 2 \text{ MeV}. \quad (1.56)$$

Apart from the suppression of general nature discussed above there is an additional suppression of $g_A(\Theta \rightarrow KN)$ due to $SU(3)$ group factors in the $\overline{\mathbf{10}} \rightarrow \mathbf{8}$ transition. This estimate has been performed assuming the chiral limit ($m_K = 0$) and zero momentum transfer in the decay. In more realistic settings the width can only go down. A recent calculation with account for the decay via higher Fock components and also for m_s corrections gives $\Gamma_\Theta = 0.7 \text{ MeV}$.³⁸

As stressed in Ref.⁵ where the narrow Θ^+ has been first predicted, in the imaginary nonrelativistic limit when ordinary baryons are made of three quarks only with no admixture of $Q\bar{Q}$ pairs the Θ^+ width tends to zero strictly.

It may seem that by the same argument all members of the exotic multiplets $(\overline{\mathbf{10}}, \frac{1}{2})$, $(\mathbf{27}, \frac{3}{2})$ etc., should be very narrow resonances but the above argument applies only to the transitions with an emission of one pseudoscalar meson. As a matter of fact it applies also to the BBV transitions where V is a vector meson that couples to baryons via the conserved vector current; such transitions are also expected to be strongly suppressed,³⁹ and the calculations³⁸ support it. However, the argument does not work for transitions with two or more pseudoscalar mesons emission. Therefore, if the phase volume allows for a decay of an exotic resonance to two or more mesons, the width does not need to be particularly narrow; it should be studied on case-to-case basis.

We now return to the Skyrme model and discuss why does it give a wide Θ^+ in the large N_c limit. As explained in Section 1, the Skyrme model is an idealization of nature: It implies that the chiral field is broad, the valence quarks are close to the negative-energy Dirac sea and cannot be separated from the sea, whereas the distortion of the sea is large. The number of $Q\bar{Q}$ pairs originating from the strongly deformed sea is $\mathcal{O}(1)$, times N_c . For example, it has been known for quite a while⁴⁰ that the fraction of nucleon spin carried by valence quarks is zero in the Skyrme model. Therefore, the Skyrme model implies the situation which is the opposite extreme from the nonrelativistic quarks where there are N_c valence quarks and zero $Q\bar{Q}$ pairs. From the viewpoint of parton distributions, both limits are discussed in some detail in Ref.¹⁷

Therefore, the Skyrme model fails to accommodate the real-world physics explaining the narrow Θ^+ , in two essential points:

- At large N_c justifying the study of the Θ^+ resonance from the kaon-Skyrmion scattering both the nucleon and the Θ^+ have an equal (and infinite) number of $Q\bar{Q}$ pairs; hence the $\Theta \rightarrow KN$ transition is not suppressed at all. [This is why we have obtained a large 90 MeV width in Section 5]
- Even if one takes a moderate $N_c = 3$, the Skyrme model implies that the $5Q$ component of the nucleon remains large, and there is no argument why the Θ^+ width should be particularly small, although it must be less than in the infinite- N_c limit.

Having this understanding in mind, in the next section we return to the Skyrme model to see if it is possible to play with its parameters in such a way that it would mimic to some extent the nonrelativistic limit. Then the “ Θ^+ ” of such a model should be narrow even if N_c is large.

1.7. Getting a narrow Θ^+ in the Skyrme model

Unfortunately, there are very few handles in the Skyrme model to play with. In fact, there are only three constants: F_π , e and the coefficient in front of the Wess–Zumino term, γ . The last one is unity in the ideal case but is not universal if, for example, $m_K \neq 0$. A general statement is that γ decreases as m_K increases. The constant F_π has to be taken 1.5 times less than its physical value to fit 940 MeV for the nucleon mass. The dimensionless coefficient e is also rather arbitrary, it is not being fixed from the experimental d -wave pion scattering. Therefore, we feel free to modify these constants at will, in order to make a theoretical point. The models we are going to present are not realistic, of course. To get more realistic models, one has either to add vector mesons²⁶ or take the Chiral Quark Soliton Model, or do something else.

1.7.1. Vanishing $m_\Theta - m_N$, vanishing Γ_Θ

We start with a simple exercise, making γ a small number. To simplify the presentation we take the chiral limit $m_K = 0$ but give the final results for $m_K \neq 0$ at the end of this subsection.

At small γ , one can compute everything analytically. In particular, one can find the regular function $g(\omega)$ (1.53) in the range of interest $\omega = k \sim \gamma$. This is done by comparing the asymptotics of the wave function (1.50) in the range $kr \ll 1$ but $r \gg r_0$ where r_0 is the coefficient in the asymptotics of the profile function $P(r) \approx r_0^2/r^2$, with the asymptotics of $\eta(r)$ at $r \gg r_0$ being

$$\eta(r) = \frac{C_1}{r^2} + C_2 r, \quad \text{from where } g(\omega) = \frac{3C_2}{C_1\omega^2}. \quad (1.57)$$

The coefficients $C_{1,2}$ are found from the following considerations. In the range of interest, $k \sim \gamma \ll 1$, the wave function $\eta(r)$ differs from the rotational wave function $\eta_{\text{rot}}(r)$ (1.49), being the exact solution of Eq. (1.48) at $\omega = 0$, by terms of the order of γ^2 . $\eta_{\text{rot}}(r)$ falls off at large distances as $r_0^2/2r^2 + 0 \cdot r$. Therefore, $C_1 = r_0^2/2 + \mathcal{O}(\gamma^2)$ whereas $C_2 = \mathcal{O}(\gamma^2)$ and can be computed, in the leading order, as a matrix element (1.46) with $\eta(r)$ substituted by $\eta_{\text{rot}}(r)$. We find

$$C_1 = \frac{r_0^2}{2} + \mathcal{O}(\gamma^2), \quad C_2 = \frac{1}{6\pi r_0^2} \left(\gamma e^2 N_c \omega - \tilde{I}_2 \omega^2 \right) = \mathcal{O}(\gamma^2) \quad (1.58)$$

where the second moment of inertia \tilde{I}_2 arises here as

$$\tilde{I}_2 = 4\pi \int dr r^2 \eta_{\text{rot}} A(r) \eta_{\text{rot}}(r).$$

The physical moment of inertia (in MeV) is obtained from the dimensionless \tilde{I}_2 as $I_2 = \tilde{I}_2 / (8e^3 F_\pi)$.

Technically, we obtain C_2 by the following trick: We integrate Eq. (1.48) multiplied to the left by $\eta_{\text{rot}}(r)$ from zero to some $r \gg r_0$, and subtract the same integral with $\eta(r)$ and $\eta_{\text{rot}}(r)$ interchanged. The terms with the first and second derivatives of η, η_{rot} become a full derivative that can be evaluated at the integration end point, while terms with no derivatives are fast convergent such that one can extend the integration range to infinity and also replace $\eta \rightarrow \eta_{\text{rot}}$ in the leading order.

The poles of the scattering amplitude are found from the equation $g(\omega) = i\omega$. Using Eqs.(1.57,1.58) we get the real and imaginary parts of the pole position:

$$\begin{aligned} \Delta &= \gamma \omega_{\text{rot}} = \mathcal{O}(\gamma), \quad \omega_{\text{rot}} = \frac{N_c}{4I_2}, \\ \Gamma &= \frac{8\pi F_\pi^2 r_0^4}{\gamma N_c} \Delta^3 = \mathcal{O}(\gamma^2) \ll \Delta, \end{aligned} \quad (1.59)$$

where $\Delta = m_\Theta - m_N$, Γ and r_0 are in physical units. These are actually the first terms in the expansion (1.55). We have developed a perturbation theory in γ and found analytically higher order terms as well. The first few orders follow the numerical determination of the pole position all the way up to $\gamma = 1$, see Fig. 10. It makes us confident that even at finite width the Θ^+ resonance is a continuous deformation of the rotational would-be zero mode.

Since in the limit of small γ everything is analytically calculable, one can check various facts. In particular, we have computed the transition axial constant $g_A(\Theta \rightarrow KN)$ from the asymptotics (1.50) of the kaon scattering wave $\eta(r)$. It gives the needed (massless) kaon pole $1/(\omega^2 - k^2)$; the axial constant is the residue of this pole, for more details see Section 8. The overall spinor coefficient ζ^α in the kaon wave is fixed from the quantization condition requesting that a state with strangeness $S = +1$ and exoticness $X = 1$ is formed.²⁵ It gives

$$\langle \zeta_\alpha^\dagger \zeta^\alpha \rangle = \frac{8}{\gamma N_c}. \quad (1.60)$$

We obtain

$$g_A(\Theta \rightarrow KN) = 8\pi \frac{F_\pi^2 r_0^2}{\sqrt{2}\gamma N_c}.$$

Given the axial constant, the pseudoscalar coupling $G(\Theta \rightarrow KN)$ can be found from the Goldberger–Treiman relation

$$G(\Theta \rightarrow KN) = \frac{m_N g_A(\Theta \rightarrow KN)}{F_\pi}.$$

The pseudoscalar coupling and the decay phase volume determines the Θ^+ width:

$$\Gamma = \frac{G^2(\Theta \rightarrow KN)\Delta^3}{4\pi m_N^2} = \frac{8\pi F_\pi^2 r_0^4}{\gamma N_c} \Delta^3.$$

Comparing it with the determination of Γ from the pole position (1.59), we see that the two ways of determining the width coincide!

We can determine the width in a third way – from the *radiation* of the kaon field by the resonance.⁴¹ According to Bohr’s correspondence principle, the quantum width is determined as the inverse time during which a resonance loses, through classical radiation, the energy difference between the neighbour states:

$$\Gamma = \frac{W}{\Delta} \quad (1.61)$$

where W is the radiation intensity, *i.e.* the energy loss per unit time. Strictly speaking, Bohr’s principle is semiclassical and applies only to the decays of the highly excited levels. In our case, however, we linearize in the kaon field at large N_c , therefore it is essentially a problem for a set of harmonic oscillators for which semiclassics is exact starting from the first excited level, here the Θ^+ .

To find the radiation intensity W , we look for the solution of Eq. (1.48) with only the outgoing wave in the asymptotics, $\eta(r) = c(kr + i)e^{ikr}/r^2$. The coefficient c is found from matching it in the range $r_0 \ll r \ll 1/k$ with the solution regular at the origin. In the leading order in γ it is the rotational mode $\eta_{\text{rot}}(r) \approx r_0^2/2r^2$; it gives $c = -ir_0^2/2$. The radiation intensity corresponding to this outgoing wave is found as the flux through a distant surface of the Poynting vector

$$T_{0i} \xrightarrow{r \rightarrow \infty} \frac{F_\pi^2}{2} (\partial_0 K_\alpha^\dagger \partial_i K^\alpha + \partial_i K_\alpha^\dagger \partial_0 K^\alpha) \xrightarrow{r \rightarrow \infty} \frac{F_\pi^2 r_0^4 k^4}{4r^2} \zeta_\alpha^\dagger \zeta^\alpha.$$

Using the normalization (1.60) we find the radiation intensity

$$W = \int d^2 S_i T_{0i} = 4\pi \frac{F_\pi^2 r_0^4 k^4}{4} \frac{8}{\gamma N_c}$$

where in the massless kaon limit $k = \omega = \Delta$. Note that the k^4 dependence is typical for the dipole radiation; it is dipole as we look for kaon radiation in p -wave. [Another characteristic feature of the dipole radiation – the $\cos^2 \theta$ angular dependence – is

not seen here because we have in fact averaged over the Θ^+ spin.] From Bohr's equation (1.61) we obtain the Θ^+ width

$$\Gamma = \frac{W}{\Delta} = \frac{8\pi F_\pi^2 r_0^4}{N_c \gamma} \Delta^3$$

again coinciding with the determination of the width from the pole position, Eq. (1.59). One can also compute the width as the inverse time during which one unit of strangeness is lost through kaon radiation, with the same result. Yet another (a 5th!) way of computing Γ – from the asymptotics of the classical profile function of the Skyrmion – will be presented in Section 9.

The derivation of Γ and Δ can be repeated for $m_K \neq 0$ in which case we find

$$\Delta = \Delta_0 \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{b m_K^2}{\Delta_0^2}} \right), \quad \Gamma = \Gamma_0 \frac{\Delta_0}{2\Delta - \Delta_0} \frac{(\Delta^2 - m_K^2)^{\frac{3}{2}}}{\Delta_0^3}, \quad (1.62)$$

where $\Delta = m_\Theta - m_N$ when $m_K \neq 0$ while the subscript 0 refers to the case of $m_K = 0$. It is remarkable that the imaginary part of the pole position Γ apparently “knows” – through unitarity – about the decay phase volume clearly visible in Eq. (1.62). The numerical coefficient b is defined as $b = 4\pi \int dr r^2 \eta_{\text{rot}}^2(r) / \tilde{I}_2 \approx 0.705$.

To conclude this subsection: in the case when the Θ^+ width is made small, we have determined it in three independent ways: *i*) from the pole position in the complex energy plane, *ii*) from the axial constant and by using the Goldberger–Treiman relation, *iii*) from the semiclassical radiation theory. All three calculations lead to the same expression for the width Γ .

1.7.2. Finite $m_\Theta - m_N$, vanishing Γ_Θ

The analytical equations of the previous subsection remain accurate as long as the imaginary part of the pole position is much less than the real part, that is insofar as $\Gamma \ll \Delta$. From Eq. (1.59) one infers that actually this condition is $\gamma e^2 N_c \ll 1$ where e^2 is the inverse coefficient in front of the Skyrme term in the action, and γ is the coefficient of the Wess–Zumino term. If one likes to fix once and forever $\gamma = 1$ (say, from topology arguments), one is still able to support the regime $\Gamma \rightarrow 0$ but $\Delta = \text{const}$ by rescaling the other two constants of the Skyrme model. Namely, we consider the following regime:

$$F_\pi = F_0 \beta^{-3}, \quad e = e_0 \beta, \quad \beta \rightarrow 0, \quad \gamma = \gamma_0 = 1. \quad (1.63)$$

The Skyrmion mass, its size and moments of inertia scale then as

$$m_N \sim \frac{F_\pi}{e} \propto \frac{1}{\beta^4} \rightarrow \infty, \quad r_0 \sim \frac{1}{F_\pi e} \propto \beta^2 \rightarrow 0, \quad I_{1,2} \sim \frac{1}{F_\pi e^3} \propto \text{const.},$$

such that the splittings between rotational $SU(3)$ multiplets remain fixed, $m_\Delta - m_N \sim \text{const.}$, and

$$\Delta = m_\Theta - m_N = \frac{N_c}{4I_2} \sim N_c F_\pi e^3 \propto \text{const.}, \quad \text{but } \Gamma_\Theta \sim N_c^2 F_\pi e^5 \propto \beta^2 \rightarrow 0. \quad (1.64)$$

We stress that Θ^+ becomes stable in this regime not because the decay phase volume tends to zero (which would have been trivial but it is not the case here) but because the Skyrmon size r_0 is small. Taking r_0 to zero we mimic to some extent the limit of nonrelativistic quarks in the Skyrme model, where we expect a narrow width. Since the Skyrme model is opposite in spirit to the nonrelativistic quarks (see Sections 1 and 6) it is difficult to achieve this limit. Indeed, the regime (1.63) is not too realistic. However it serves well to illustrate the point: When Γ is small, the real part of the pole position coincides with the rotational frequency

$$m_\Theta - m_N = \omega_{\text{rot}} = \frac{N_c}{4I_2} \quad (1.65)$$

as it follows from the quantization of the $SU(3)$ rotations, Eq. (1.24).

Is it a coincidence? Probably, not: Θ^+ is an $SU(3)$ rotational excitation of the nucleon. (At large N_c the rotation is more like a precession near the “North pole” but nevertheless.) It remains a (deformed) rotational state even in the worse case scenario provided by the Skyrme model where at “realistic” parameters it becomes a broad and hence strong resonance but then what should be called the resonance mass becomes ambiguous. Its precise determination is then from the pole position which is away from the real axis, such that the real part of the pole position does not need to coincide with the rotational splitting just for the trivial reason that the imaginary part is large.

Therefore, the key issue is the resonance width. On the one hand, a rotating body must radiate, in this case the kaon field. Since in the $\frac{1}{2}^+ \rightarrow \frac{1}{2}^+$ transition the kaon is in the p wave, the Θ^+ width is entirely due to dipole radiation. The dipole radiation intensity is proportional to $(\ddot{d})^2 = \omega^4 d^2$ at small frequencies where d is the dipole moment. Generically, $d \sim r_0$ where r_0 is the characteristic size of the system. The Skyrme model illustrates the generic case, therefore the only way to suppress the dipole radiation at fixed $\omega = \Delta$ is to shrink the size r_0 as we have done above. On the other hand, in our case it is the *transition* dipole moment d corresponding to strangeness emission, which can be, in principle, much less than r_0 , even zero. We have argued in Section 6 that in the real world the transition dipole moment is small as the nucleon is essentially nonrelativistic and hence has a small $5Q$ component. If the width is small, we see no reasons why would not the real part of the Θ^+ pole coincide with the rotational splitting of the $SU(3)$ multiplets^a.

^aA quantum-mechanical counter-example by T. Cohen⁴² does not seem to capture the necessary physics as the spectrum there is discrete and there is no radiation.

1.8. Goldberger–Treiman relation and the Θ^+ width

In this section we show that the Θ^+ width can be expressed through the transition axial constant *provided* the width is small. We reaffirm the validity of a modified Goldberger–Treiman relation between the axial and pseudoscalar ΘKN constants in the chiral limit. We derive these relations in the framework of the Skyrme model where all equations are explicit. However, these relations are, of course, of a general nature.

If Θ^+ is a narrow and well-defined state one can define the transitional axial $g_A = g_A(\Theta \rightarrow KN)$ and pseudoscalar $G = G(\Theta \rightarrow KN)$ constants as

$$\frac{\varepsilon_{\alpha\beta}}{\sqrt{2}} \langle \Theta_k^+ | j_{\mu 5}^\alpha(x) | N^{\beta,i} \rangle = e^{iq \cdot x} \bar{u}(\Theta, k) \left(\frac{\gamma_\mu}{2} g_A - \frac{q_\mu}{q^2} F_\pi G \right) \gamma_5 u(N, i) \quad (1.66)$$

where $i, k = 1, 2$ are the nucleon and Θ^+ spin projections, $\alpha, \beta = 1, 2$ are the isospin projections of the nucleon and of the kaon current; we are interested in the isospin $T=0$ channel. Finally, u, \bar{u} are N and Θ^+ 4-spinors. We assume that they obey the non-relativistic normalization $\bar{u}(i)u(k) = \delta_{ik}$. In the non-relativistic limit appropriate at large N_c one has:

$$\bar{u}(k) \gamma_5 u(i) = \psi^*(k) \frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{2m} \psi(i) = \frac{1}{2m} (\mathbf{q} \cdot \boldsymbol{\sigma})_k^i \quad (1.67)$$

where $\psi(i)$ is a non-relativistic 2-spinor with polarization i .

The modified Goldberger–Treiman relation follows immediately from the conservation of the axial current, $\partial_\mu j_{\mu 5}^\alpha = 0$:

$$g_A(\Theta \rightarrow KN)(m_N + m_\Theta) = 2G(\Theta \rightarrow KN)F_\pi.$$

It should be stressed that it holds true even if m_Θ differs significantly from m_N . In the large N_c limit however one can put $m_\Theta \approx m_N$.

Let us consider now the nucleon matrix element of the product of two strangeness-changing axial currents $j_{\mu 5}^\alpha(x)$ and expand it in intermediate states $|n\rangle$:

$$\begin{aligned} \Pi^{T=0}(\omega, \mathbf{q})_k^i &= \int d^4x e^{iq \cdot x} \frac{\varepsilon_{\alpha_1 \beta_1}}{\sqrt{2}} \langle N_{\beta_1 k} | j_{\nu 5}^{\dagger \alpha_1}(x) j_{\mu 5}^{\alpha_2}(0) | N^{\beta_2 i} \rangle \frac{\varepsilon_{\alpha_2 \beta_2}}{\sqrt{2}} \\ &= \sum_n \frac{\varepsilon_{\alpha_1 \beta_1}}{\sqrt{2}} \langle N_{\beta_1 k} | j_{\nu 5}^{\dagger \alpha_1}(0) | n(\mathbf{q}) \rangle 2\pi \delta(\omega - E_n(\mathbf{q})) \langle n(\mathbf{q}) | j_{\mu 5}^{\alpha_2}(0) | N^{\beta_2 i} \rangle \frac{\varepsilon_{\alpha_2 \beta_2}}{\sqrt{2}}. \end{aligned} \quad (1.68)$$

Here $E_n(\mathbf{q})$ is the kinetic energy of the intermediate state. Since the nucleon is infinitely heavy at large N_c the energy ω and the 3-momentum \mathbf{q} are conserved. We write relativistic equations for the kaon field, however.

The correlation function (1.68) can be calculated *e.g.* in the Skyrme model. We wish to isolate the kaon pole contribution to the strange axial current, that is we have to consider $\omega^2 \approx \mathbf{q}^2 + m_K^2$ where we temporarily take the chiral limit, $m_K = 0$, for the current to be conserved. In fact, this requirement can be relaxed.

The singular contribution to the current arises from the asymptotics of the kaon scattering wave (1.50):

$$j_{\mu 5}^{\alpha}(\omega, \mathbf{q}) = iq_{\mu} F_{\pi} K^{\alpha}(\omega, \mathbf{q}) = -q_{\mu} F_{\pi} \frac{\sin \delta(\omega)}{\omega^2 - \mathbf{q}^2} (\mathbf{q} \cdot \boldsymbol{\tau})_{\beta}^{\alpha} \mathbf{b}^{\dagger \beta}(\omega) \sqrt{\frac{4\pi}{\omega^3}}. \quad (1.69)$$

The last factor arises here in accordance with the commutation relation for the creation-annihilation operators $[\mathbf{b}^{\dagger \beta}(\omega_1) \mathbf{b}_{\alpha}(\omega_2)] = 2\pi \delta_{\alpha}^{\beta} \delta(\omega_1 - \omega_2)$.²⁵ Substituting Eq. (1.69) into Eq. (1.68) we obtain

$$\Pi_{\mu\nu}^{T=0}(\omega, \mathbf{q})_k^i = q_{\mu} q_{\nu} \frac{4\pi F_{\pi}^2}{\omega^3} \mathbf{q}^2 \delta_k^i \frac{\sin^2 \delta(\omega)}{(\omega^2 - \mathbf{q}^2)^2}. \quad (1.70)$$

The correlation function is therefore expressed through the phase shift $\delta(\omega)$! The conservation of the axial current implies that there is also a contact term in the correlation function, proportional to $g_{\mu\nu}$; the coefficient in front of it must be exactly the coefficient in front of $q_{\mu} q_{\nu} / q^2$, with the minus sign.

Let us now assume that one of the intermediate states in Eq. (1.68) is a *narrow* Θ^+ resonance. Then, on the one hand, at $\omega \approx \Delta$ the phase shift $\delta(\omega)$ must exhibit the Breit–Wigner behaviour as it follows from unitarity:

$$\sin^2 \delta(\omega) = \frac{\Gamma^2/4}{(\omega - \Delta)^2 + \frac{\Gamma^2}{4}} \xrightarrow{\Gamma \rightarrow 0} \frac{\Gamma}{4} 2\pi \delta(\omega - \Delta). \quad (1.71)$$

On the other hand, one can extract the contribution of the Θ^+ intermediate state using the definition of the matrix elements of the axial current (1.66). Taking there the contribution that is singular near the kaon pole and recalling Eq. (1.67) we get

$$\Pi_{\mu\nu}^{T=0}(\omega, \mathbf{q})_k^i = 2\pi \delta(\omega - \Delta) \frac{q_{\mu} q_{\nu}}{(q^2)^2} F_{\pi}^2 G^2 \frac{\mathbf{q}^2}{4m_N^2} \delta_k^i. \quad (1.72)$$

We now compare Eq. (1.72) and Eqs. (1.70, 1.71) and immediately obtain the already familiar equation for the Θ^+ width through the ΘKN pseudoscalar coupling G (cf. Section 7):

$$\Gamma = \frac{G^2(\Theta \rightarrow KN)}{4\pi m_N^2} \mathbf{p}^3 \quad (1.73)$$

where \mathbf{p} is the kaon momentum, equal to Δ in the chiral limit. We stress that the Born graph for the KN scattering with pseudoscalar Yukawa coupling arises automatically – through unitarity – from the KN scattering phase, provided it corresponds to a narrow resonance.

To conclude, if Θ^+ happens to be a narrow resonance, one can find its width from the ΘKN transition axial coupling or, thanks to the Goldberger–Treiman relation, from the transition pseudoscalar coupling (it is contrary to the recent claim of Ref.^{29,43}). This is how the narrow Θ^+ has been first predicted⁵ and how a more stringent estimate of the width $\Gamma_{\Theta} \sim 1$ MeV has been recently performed.^{16,37,38}

1.9. Finite- N_c effects in the Θ^+ width

In any chiral soliton model of baryons, the baryon-baryon-meson coupling can be written in terms of the rotational coordinates given by the $SU(3)$ matrix R as²⁸

$$L = -ip_i \frac{3G_0}{2m_N} \frac{1}{2} \text{Tr}(R^\dagger \lambda^a R \tau^i) \quad (1.74)$$

where λ^a is the Gell-Mann matrix for the pseudoscalar meson of flavour a , and p_i is its 3-momentum. The pseudoscalar coupling G_0 is directly related to the asymptotics of the Skyrme profile function $P(r) \approx r_0^2/r^2$ ²⁸:

$$G_0 = \frac{8\pi}{3} F_\pi m_N r_0^2. \quad (1.75)$$

Note that $G_0 = \mathcal{O}(N_c^{\frac{3}{2}})$. In a generic case there are baryon-baryon-meson couplings other than (1.74), labeled in Ref.⁵ by G_1 and G_2 . It is the interplay of these constants that leads to a small Θ^+ width. In the nonrelativistic limit the combination of $G_{0,1,2}$ is such that the width goes to zero strictly, however each of the constants remain finite being then determined solely by valence quarks. Unfortunately, in the Skyrme model $G_{1,2}$ are altogether absent, related to the fact that there are no valence quarks in the Skyrme model. For example, G_2 is proportional to the fraction of nucleon spin carried by valence quarks which is known to be exactly zero in the Skyrme model.⁴⁰ Since only the coupling G_0 is present in the Skyrme model, we are forced to mimic the nonrelativistic limit there by taking the size r_0 to zero, which leads to unrealistic parameters. In any chiral model with explicit valence quarks there are less traumatic ways to obtain a very small Θ^+ width.

In the chiral limit $SU(3)$ symmetry is exact, therefore Eqs.(1.74,1.75) determine also the leading term in the $\Theta \rightarrow KN$ decay width, provided Θ^+ is understood as an excited rotational state of a nucleon.⁵ For arbitrary N_c the appropriate Clebsch-Gordan coefficient has been computed by Praszalowicz:⁴⁴

$$\Gamma_\Theta(N_c) = \frac{3(N_c + 1)}{(N_c + 3)(N_c + 7)} \frac{3}{8\pi m_N^2} G_0^2 |\mathbf{p}|^3. \quad (1.76)$$

To compare it with the width computed in Section 7 from the imaginary part of the pole in the kaon-Skyrmion scattering amplitude, one has to take the limit $N_c \rightarrow \infty$, as only in this limit the use of the Callan-Klebanov linearized scattering approach is legal. Using (1.75) we find

$$\Gamma_\Theta(N_c \rightarrow \infty) = \frac{8\pi F_\pi^2 r_0^4}{N_c} \Delta^3 = \mathcal{O}(N_c^0), \quad (1.77)$$

which coincides exactly with the width obtained in Section 7 by other methods, in particular from the resonance pole position, where one has to put the coefficient $\gamma=1$. To guarantee the validity of this result one has to make sure that the width is small, $\Gamma \ll \Delta$, for example, by taking the limit considered in subsection 7.2. In more realistic models the condition $\Gamma \ll \Delta$ can be achieved not by taking small G_0

but due to the cancelation of several pseudoscalar coupling $G_{0,1,2}$ as it in fact must happen in the nonrelativistic limit. Then, as shown from unitarity in Section 8, Eq. (1.76) modified to incorporate other couplings^{5,44} remains valid.

Looking into Eq. (1.76) we can discuss what happens with the width as one goes from the idealized case of $N_c = \infty$ to the real world with $N_c = 3$. Unfortunately, at finite N_c the whole Skyrmon approach becomes problematic since quantum corrections to the saddle point are then not small. Quantum corrections to a saddle point in general and here in particular are of two kinds: coming from zero and nonzero modes. Corrections from nonzero modes can be viewed as a meson loop in the Skyrmon background. As any other quantum loop in 4 dimensions, it has a typical additional suppression by $1/(2\pi)$ arising from the integral over loop momenta $\int d^4p/(2\pi)^4$. We remind the reader that in QED radiative corrections are not of the order of α , the fine structure constant, but rather $\alpha/(2\pi) \approx 10^{-3}$. Therefore, quantum corrections from nonzero modes are expected to be of the order of $1/(2\pi N_c) \approx 1/20$ and look as if they can be neglected. As to zero modes, which are the translations and the rotations of the Skyrmon as a whole, they do not lead to the additional $1/(2\pi)$ suppression. On the contrary, they lead to “kinematical” factors like the Clebsch–Gordan coefficient in Eq. (1.76), which bear huge $1/N_c$ corrections. Hence it is desirable to take rotations into account exactly for any N_c .

We are therefore inclined to take Eq. (1.76) at face value for any N_c and claim that it is the leading effect in accounting for finiteness of N_c . At $N_c=3$ it leads to the relation

$$\Gamma_{\Theta}(N_c=3) = \frac{1}{5} \Gamma_{\Theta}(N_c \rightarrow \infty). \quad (1.78)$$

The Clebsch–Gordan coefficient “1/5” was actually used in the original paper⁵ predicting a narrow pentaquark. A large suppression of Γ_{Θ} as compared to its asymptotic value at $N_c = \infty$ has been also noticed in Ref.²⁹ in another estimate of the finite N_c effects. Whatever is the width found from the pole position in the kaon-Skyrmion scattering amplitude, the real Θ^+ width is expected to be at least 5 times less! Estimates for the real-world $N_c=3$ in Refs.^{16,37,38} demonstrate that it can easily be obtained at the level of 1 MeV or even less, without any fitting parameters.

1.10. Conclusions

The remarkable idea of Skyrme that baryons can be viewed as nonlinear solitons of the pion field, finds a justification from the modern QCD point of view. However, the concrete realization of this idea – the use of the two- and four-derivative Skyrme Lagrangian supplemented by the four-derivative Wess–Zumino term – is an oversimplification of reality. Therefore, the Skyrme model as it is, may work reasonably well for certain baryon observables but may fail qualitatively for other.

To understand where the Skyrme model fails, one has to keep in mind that the

model implies that the valence quarks are close to the negative-energy Dirac sea and cannot be separated from the sea that is strongly distorted. The number of $Q\bar{Q}$ pairs in a baryon, corresponding to a strongly polarized sea is $\mathcal{O}(1)$, times N_c . Exotic baryons are then not distinguishable from ordinary ones as they differ only by one additional $Q\bar{Q}$ pair as compared to the infinite $\mathcal{O}(N_c)$ number of pairs already present in the nucleon in that model, hence the exotic decays are not suppressed. In principle, it does not contradict QCD at strong coupling, however in reality we know that the octet and decuplet baryons are mainly ‘made of’ $N_c=3$ constituent quarks with only a small (order of 30%) admixture of the $N_c+2=5$ quark Fock component. In fact there is an implicit small parameter in baryon physics that may be called “relativism” $\epsilon \ll 1$ such that valence quark velocities are $v^2/c^2 \sim \epsilon$ and the number of $Q\bar{Q}$ pairs is ϵN_c .¹⁶ For observables where the “nonrelativism” is essential one expects a qualitative disagreement with the Skyrme model predictions. For computing such observables it is better, while preserving the general and correct Skyrme’s idea, to use a model that interpolates between the two extremes: the Skyrme model and the nonrelativistic quark model where there are no antiquarks at all.

Quantization of the $SU(3)$ zero rotational modes of the Skyrmion, whatever is its dynamical realization, leads to the spectrum of baryons forming a sequence of bands: each band is characterized by “exoticness”, *i.e.* the number of additional $Q\bar{Q}$ pairs minimally needed to form a baryon multiplet. Inside the band, the splittings are $\mathcal{O}(1/N_c)$ whereas the splittings between bands with increasing exoticness is $\mathcal{O}(1)$, see Fig. 5. At large N_c the lowest-mass baryons with nonzero exoticness (like the Θ^+ baryon) have rotational wave functions corresponding to a small-angle precession. Therefore, the Θ^+ and other exotic baryons can be, at asymptotically large N_c , studied *à la* Callan–Klebanov by considering the small oscillations of the kaon field about a Skyrmion, or the kaon-Skyrmion scattering in the linear order.

This problem has been solved numerically by Klebanov *et al.*²⁷ who have found that there is no resonance or bound state with the Θ^+ exotic quantum numbers at least in the large N_c limit, and suggested that it therefore could be an artifact of the rigid rotator approximation. In this paper, we study this scattering in more detail and come to the opposite conclusion. While reproducing numerically the phase shifts found in²⁷ we find, both analytically and numerically, that there is a pole in the complex energy plane, corresponding to a strong Θ^+ resonance which would have definitely revealed itself in KN scattering. Moreover, its origin is precisely the $SU(3)$ rotational mode. By varying the Skyrme model parameters, we are able to make Θ^+ as narrow as one likes, as compared to the resonance excitation energy which can be hold arbitrary. Being arbitrary it nevertheless coincides with the rotational excitation energy. To understand better the origin of the Θ^+ width, we have computed it in five different ways yielding the same result. The problem is not the existence of Θ^+ which *is* predicted by the Skyrme model and is a rotational excitation there, but what dynamics makes it narrow.

Although we can deform the parameters of the Skyrme model to make a finite-energy Θ^+ narrow, they are not natural. It is precisely a problem where the deficiency of the Skyrme model mentioned above becomes, unfortunately, critical. To get a chance of explaining the narrow width, one needs a model that interpolates between the Skyrme model and the nonrelativistic quarks models. The narrow Θ^+ is near the nonrelativistic end of this interpolation. Fortunately, the Chiral Quark Soliton Model makes the job and indeed estimates of the Θ^+ width there appear naturally with no parameter fitting at the 1 MeV level.

It is exciting and challenging to write this paper at the time when experimental evidence in favour of the exotic pentaquark Θ^+ is still controversial. We hope that we have waived certain theoretical prejudice against Θ 's existence and its small width, so it must be there.

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